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ON CHIRAL P-FORMS

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Abstract

Some aspects of chiral p-forms, in particular the obstruction that makes it hard to define covariant Green functions, are discussed. It is shown that a proposed resolution involving an infinite set of gauge fields can be made mathematically rigorous in the classical case. We also give a brief demonstration of species doubling for chiral bosons.

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1. INTRODUCTION

What does it mean to say that a theory is "relativistic"? As a matter of fact two *a priori* different meanings of the term "relativistic invariance" are in use [1]. The first demands that there should be no preferred reference frame. Technically, the main requirement here is that there should exist a frame independent prescription for how to transform a description of a physical situation that uses a particular inertial frame to a description based on another inertial frame, and that the set of such transformations have to obey the multiplication laws of the Poincaré group. This is of course enough to make a search for scalar quantities meaningful. If the description of the system is based on the principle of least action, it is natural to demand also that the transformations should be implementable as canonical or unitary transformations.

Another, and stronger, meaning of the term "relativistic invariance" requires that the description of the physical system should be "manifestly covariant", that is to say based entirely on finite dimensional tensors (and spinors). This demand is not only made for reasons of practical convenience, but also for a deeper philosophical reason: It ensures that the theory can be formulated in a way that is entirely independent of reference frames.

Note that, throughout, we are concerned only with the special theory of relativity.

Now it is natural to ask: how much stronger than the first is the second definition of "relativistic invariance"? One might think that it is much stronger, or alternatively one might think that the two definitions are equivalent. In four spacetime dimensions, if we stick to the kind of models that can be described by field theories and restrict our attention to non-interacting systems, the latter answer can be shown to be correct by means of explicit constructions. Indeed it is known that to every irreducible representation of the Poincaré group (except the continuous spin representations, which we ignore) there corresponds a manifestly covariant free field theory whose solution space carries the representation in question [2]. Curiously, the situation in hypothetical spacetimes of higher dimensions is less clear. Marcus and Schwarz [3] made the remarkable claim that chiral p-forms [4] — a kind of fields that can occur in space-times of twice odd dimension — do not admit a manifestly covariant description, although they are relativistically invariant according to the first definition. Consequently the two possible meanings of "relativistic invariance" would not be equivalent in such spacetimes. The direct relevance to physics of this counterexample is not obvious, although chiral p-forms do enter in the spectrum of type IIB superstrings, so that they occupy a small corner of a serious theory which may be of physical relevance. (A string that couples directly to a chiral p-form was constructed recently [5].) Be that as it may, our interest here is entirely in the questions of principle that the existence of this model raises. The principles are of physical relevance, and we wish to understand them.

A chiral p-form has a number of properties which are very peculiar. Except for the fact that its excitations obey Bose statistics, it shares many of the features of fermionic fields: it obeys a field equation which is linear in derivatives, it leads to species doubling if one attempts to define it on a lattice [6], and it gives rise to gravitational anomalies [7]. A literature on the subject already exists — and an even larger number of papers have

been written on the simplest case, chiral bosons in $1 + 1$ dimensions, which from our point of view is a degenerate case since it leads to no difficulties with respect to covariance. The main reason why we intend to return to the subject in the present paper is simply that we believe that we can present some of the issues in a way that is somewhat more clear than that of previous treatments. Some of our observations are new, however.

In section 2 of this paper we try to explain the sense in which chiral p-forms violate manifest covariance. As we will emphasize, it is not enough to find a covariant action. One must also write down covariant Green functions, which is a harder task. We use the Hamiltonian formulation of the model to explain why it is hard, since it is enough to study the equal time restrictions for this purpose. The conclusion is that manifest covariance is violated in the same sense as manifest covariance is violated by non-covariant gauges (such as the Coulomb gauge) in electrodynamics. In section 3 we discuss a proposed solution of the problem which involves an infinite set of auxiliary fields [8], and we give a rigorous version of this proposal in the classical, $1 + 1$ dimensional case. In section 4 we turn to species doubling. Concentrating on chiral bosons in two spacetime dimensions, we show - in a way that we find more direct than the original demonstration [6] - precisely why species doubling occurs. Section 5 summarizes open problems.

2. MANIFEST COVARIANCE

Let us assume that the space-time dimension D is even, but twice odd, so that

$$D = 2n = 2(p + 1) , \quad (1)$$

where p is an even number. We also assume that we are in flat Minkowski spacetime, since nothing is gained by considering a more general situation. Then we define a p-form, that is to say a totally antisymmetric tensor field $A_{\alpha_1 \dots \alpha_p}$ with p indices, and we also introduce the notation

$$A_{\alpha[p]} \equiv A_{\alpha_1 \dots \alpha_p} . \quad (2)$$

Thus $\alpha[p]$ stands for an antisymmetric collection of p indices. Further developments of this notation, such as how to handle index contractions, should be self evident. We will use it when it is convenient to do so.

Next we introduce the field strength

$$F_{\alpha[n]} = n! \partial_{[\alpha_1} A_{\alpha_2 \dots \alpha_n]} , \quad (3)$$

where the square brackets denote anti-symmetrization with weight one. There is an action for such objects that is a natural generalization of the Maxwell action, namely

$$S[A] = -\frac{1}{2n!} \int d^{2n}x F_{\alpha[n]} F^{\alpha[n]} . \quad (4)$$

This defines the dynamics of an ordinary p-form. Next we define the Hodge dual

$$\star F_{\alpha[n]} = \frac{1}{n! \sqrt{-g}} g_{\alpha_1 \beta_1} \dots g_{\alpha_n \beta_n} \epsilon^{\beta[n] \gamma[n]} F_{\gamma[n]} . \quad (5)$$

Since n is odd, and

$$\star \star F_{\alpha[n]} = F_{\alpha[n]} \quad (6)$$

in Minkowski space, the eigenvectors of the \star -operator are real. Therefore the self-dual part of the field strength, which obeys

$$\star F_{\alpha[n]} = F_{\alpha[n]} , \quad (7)$$

is real. This is an important difference between twice odd and twice even dimensions. There is another difference that is even more relevant to us, and we will come to it soon.

The dynamics of a chiral p-form is defined by eq. (7), which from now on we will adopt as our field equation. Solutions of this equation automatically obey the field equations derived from the action (4). The simplest example of the self-duality condition occurs in $D = 1 + 1$ dimensions, where the potential is a scalar;

$$F_\alpha = \partial_\alpha \phi . \quad (8)$$

Then the self-duality condition (7) becomes the equation for a chiral boson. Indeed, writing it out in components we find

$$\star F_\alpha = F_\alpha \quad \Leftrightarrow \quad \dot{\phi} = \phi' , \quad (9)$$

where the dot and the prime denote time and space derivatives, respectively. It is often convenient to keep this example in mind when following the calculations for higher dimensions.

So far, everything is manifestly covariant in arbitrary D . The difficulty first appears when one attempts to derive the self-duality condition from an action. Actually it is not so easy to find an action, unless one first casts the equations in Hamiltonian form [9] [10]. Naturally, the Hamiltonian formulation itself violates manifest covariance in a sense. However, this happens in a perfectly controlled way, and there is therefore no reason why one should not use the Hamiltonian formulation to discuss the issue of manifest covariance [1]. Within the Hamiltonian formulation we will see the problem when we try to write down the Poisson brackets — that is to say the equal time restriction of the commutator Green function. In the Hamiltonian as well as in the Lagrangian formulation, manifest covariance is violated when one inverts a certain “matrix”, to define respectively the Poisson brackets and the Green functions. A manifestly covariant Lagrangian is simply not enough.

We begin, then, with the phase space version of the action for an ordinary p-form:

$$S = \int d^{2n}x \left[\dot{A}_{a[p]} E_{a[p]} - \frac{1}{2p!} (E_{a[p]} E_{a[p]} + B_{a[p]} B_{a[p]}) - p \Lambda_{a[p-1]t} \partial_{a_1} E_{a_1 a[p-1]} \right] . \quad (10)$$

Here the “electric” field $E_{a[p]}$ is the momentum canonically conjugate to the p-form, and $B_{a[p]}$ is the “magnetic” field. Explicitly,

$$E_{a[p]} = F_{ta[p]} \quad B_{a[p]} \equiv \frac{1}{n!} \epsilon_{a[p]b[n]} F_{b[n]} . \quad (11)$$

Also the time-space component $A_{ta[p-1]}$ enters the action in the guise of the Lagrange multiplier $\Lambda_{a[p-1]}$, giving rise to a first class constraint, which is analogous to Gauss’ law for the Maxwell theory. We wish to constrain this model further, so that only self-dual solutions occur. In canonical language, eq. (7) becomes

$$E_{a[p]} = B_{a[p]} . \quad (12)$$

It is straightforward to amend the phase space action so that it gives rise to this constraint: from now on we consider

$$S = \int d^{2n}x [\dot{A}_{a[p]} E_{a[p]} - \frac{1}{2p!} (E_{a[p]} E_{a[p]} + B_{a[p]} B_{a[p]}) - \Lambda_{a[p]} (E_{a[p]} - B_{a[p]})] . \quad (13)$$

The only change is in the Lagrange multiplier. The constraint is given by eq. (12).

We now come to the second crucial difference between space-times whose dimension are twice odd or twice even. In the twice odd case, the self-duality constraint is partially second class (in Dirac’s terminology [11]) , since

$$\begin{aligned} \{E_{a[p]}(x) - B_{a[p]}(x), E_{b[p]}(y) - B_{b[p]}(y)\} = \\ = -(1 + (-1)^p) \epsilon_{a[p]cb[p]} \partial_c \delta(x, y) = -2 \epsilon_{a[p]b[p]c} \partial_c \delta(x, y) . \end{aligned} \quad (14)$$

By contrast, in the twice even case p is odd, the right hand side vanishes, and all the constraints are first class, which means that a new gauge symmetry appears. There will then be no local degrees of freedom left in the model. Returning to the twice odd case, it is crucial to observe that the self-duality constraint contains both second and first class constraints. It contains second class constraints because the right hand side of eq. (14) is non-zero, but it also contains first class constraints, because the divergence of the “magnetic” field is identically zero, so that eq. (12) implies “Gauss’ law”. Gauss’ law is first class and remains as a generator of gauge transformations.

In the twice odd case the Hamiltonian is non-vanishing also when the self-duality constraint is imposed (whereas the Hamiltonian is zero for self-dual configurations in twice even dimensions). We can summarize the above discussion by the weak equality

$$H \approx \frac{1}{p!} \int d^{n+p}x B_{a[p]} B_{a[p]} , \quad (15)$$

with the understanding that we have to take into account the constraints

$$\Phi_{a[p]} \equiv E_{a[p]} - B_{a[p]} \approx 0 , \quad (16)$$

where $E_{a[p]}$ is the “naive” canonical momentum.

The next step, which has to be taken before we can say that we have defined a consistent Hamiltonian system, is to solve the second class constraints. To do this it is necessary to isolate the second class part of eq. (16). There is no unique way to do this, but all ways entail a violation of manifest covariance. A possibility that springs immediately to mind is to introduce the transverse and longitudinal projection operators

$$\Pi_{ab}^T \equiv \delta_{ab} - \frac{1}{\Delta} \partial_a \partial_b \quad \Pi_{ab}^L \equiv \frac{1}{\Delta} \partial_a \partial_b , \quad (17)$$

where Δ is the Laplacian. Then we can decompose

$$\begin{aligned} A_{a[p]} &= A_{a[p]}^T + A_{a[p]}^L , \\ A_{a[p]}^T &\equiv \Pi_{a_1 b_1}^T \dots \Pi_{a_p b_p}^T A_{b[p]} , \end{aligned} \quad (18)$$

and similarly for all other objects. We then see that the transverse part of the constraint is second class, and the longitudinal part first class:

$$\begin{aligned} \{\Phi_{a[p]}^T(x), \Phi_{b[p]}^T(y)\} &= -2\epsilon_{a[p]b[p]c} \partial_c \delta(x, y) \\ \{\Phi_{a[p]}^L(x), \Phi_{b[p]}^L(y)\} &= 0 . \end{aligned} \quad (19)$$

We can now solve the second class constraints for the transverse “electric” field, and compute the resulting Dirac brackets for the transverse vector potential. They are

$$\{A_{a[p]}^T(x), A_{b[p]}^T(y)\} = \frac{1}{2p!} \epsilon_{a[p]b[p]c} \frac{1}{\Delta} \partial_c \delta(x, y) . \quad (20)$$

Now we have arrived at a consistent Hamiltonian system, described by the Hamiltonian (15), the Poisson brackets (20), the naive Poisson brackets for the longitudinal parts of the original dynamical variables, and the first class constraints $\Phi_{a[p]}^L \approx 0$. No gauge fixing has been performed, and yet our formulæ are non-local and they violate manifest covariance in exactly the same sense that the Coulomb gauge in electrodynamics violates manifest covariance. The violation is caused by the non-covariant and non-local Poisson brackets.

When $D = 1 + 1$ the transverse part of the potential is missing. Therefore — as noted by Marcus and Schwarz [3] — there is no problem with covariance in this case. The only remaining complication is that the boundary conditions have to be chosen such that the Dirac brackets are well defined.

Although no gauge has been fixed, there was an element of choice in the above discussion. The set of constraints $\Phi_{a[p]}$ that we had at the outset can be decomposed into first

and second class constraints in other ways. For instance, if we let the index i range from 1 to $D - 2$, and use z to denote the remaining spatial coordinate, we find a subset of the constraints which obeys

$$\{\Phi_{i[p]}(x), \Phi_{j[p]}(y)\} = -2\epsilon_{i[p]j[p]}\partial_z\delta(x, y) . \quad (21)$$

This subset of the constraints can be declared to be second class, and once the Dirac brackets have been computed one sees that the remaining constraints are first class. We arrive at a formulation which differs from the first, but again manifest covariance is violated — this time in exactly the same sense that the axial gauge in electrodynamics violates manifest covariance. There is simply no covariant way to define the symplectic structure that we are studying, and by implication the same statement is true for the Green functions.

This violation of manifest covariance is rather mild, however. The gauge invariance that is also present in the model selects a set of gauge invariant observables, namely the “magnetic” fields $B_{a[p]}$. The Dirac bracket

$$\{B_{a[p]}(x), B_{b[p]}(y)\} = -\epsilon_{a[p]b[p]c}\partial_c\delta(x, y) \quad (22)$$

is perfectly local, and the energy-momentum densities — which can be expressed entirely in terms of $B_{a[p]}$ — transforms this field in a covariant way. (This point was stressed by Henneaux and Teitelboim [10].) This is again similar to electrodynamics in the Coulomb gauge.

3. A COVARIANT ACTION

An obvious Lagrangian for chiral p-forms is obtained by performing a Legendre transformation of the Hamiltonian that we have studied. To do this it is convenient to split the Lagrange multiplier $\Lambda_{a[p]}$ into transverse and longitudinal parts, and use the longitudinal part as the time-space component of the potential. In this way we can rewrite the phase space action (13) as

$$S = \frac{1}{p!} \int d^{2n}x [F_{ta[p]}E_{a[p]} - \frac{1}{2}(E_{a[p]}E_{a[p]} + B_{a[p]}B_{a[p]}) - p!\Lambda_{a[p]}^T(E_{a[p]} - B_{a[p]})] . \quad (23)$$

If we integrate out first $E_{a[p]}$ and then $\Lambda_{a[p]}^T$ we obtain a Lagrangian which is, however, not manifestly covariant. Thus the naive attempt to derive a manifestly covariant Lagrangian does not work.

Is it possible to do better? Indeed Siegel [12] quickly found a covariant Lagrangian, which consists of the ordinary action for a p-form with a certain cubic term added. A correct analysis [9] shows that Siegel’s covariant action leads to precisely the same Hamiltonian formulation as the one we have studied. Therefore Siegel’s action is covariant, but

it can not be used to derive covariant Green functions. At this point then, the claim by Marcus and Schwarz still stands. Incidentally, this is quite similar to the situation for the Green-Schwarz superstring [13], although the latter is not a field theory action, and moreover it has at least one alternative, fully covariant description, namely the NSR formalism.

However, some years later a manifestly covariant formulation of chiral p-forms was in fact presented [8]. (Actually the original reference treats only the two dimensional case, but the generalisation to higher dimensions is straightforward [14].) An awkward feature of this formulation is that it requires an infinite set of auxiliary fields. The phase space action is

$$S = \sum_{n=0}^{\infty} \int d^6x \left[\dot{A}_{ab}^{(n)} E_{ab}^{(n)} - \frac{(-1)^n}{4} (E_{ab}^{(n)} E_{ab}^{(n)} + B_{ab}^{(n)} B_{ab}^{(n)}) + \Lambda_{ab}^{(n+1)} \Psi_{ab}^{(n+1)} - 2A_{ta}^{(n)} \mathcal{G}_a^{(n)} \right], \quad (24)$$

where the action depends on an infinite set of fields and Lagrange multipliers labelled by the index n , and we have specified $D = 5 + 1$ for definiteness. The constraints are

$$\mathcal{G}_a^{(n)} = \partial_b E_{ab}^{(n)} \approx 0 \quad (25)$$

$$\Psi_{ab}^{(n+1)} = E_{ab}^{(n)} - B_{ab}^{(n)} + E_{ab}^{(n+1)} + B_{ab}^{(n+1)} \approx 0. \quad (26)$$

Note that this time the constraints $\Psi^{(n)} \approx 0$ do not imply that Gauss' law holds, and therefore the latter must be explicitly added. It is easy to check that the Hamiltonian is weakly gauge invariant and that all the constraints are first class. The corresponding Lagrangian is manifestly covariant:

$$S = \sum_{n=0}^{\infty} \int d^6x \frac{(-1)^n}{3} \left(-\frac{1}{4} F_{\alpha\beta\gamma}^{(n)} F^{\alpha\beta\gamma}_{(n)} - \Lambda_{(n+1)}^{\alpha\beta\gamma} (F_{\alpha\beta\gamma}^{(n)} - F_{\alpha\beta\gamma}^{(n+1)}) + \Lambda_{(n+1)}^{\alpha\beta\gamma} \Lambda_{\alpha\beta\gamma}^{(n+2)} \right). \quad (27)$$

Here we have set

$$\Lambda_{ab} \equiv \Lambda_{tab} \quad (28)$$

and the Lagrange multiplier fields are alternately self-dual and anti-self-dual;

$$\star \Lambda_{\alpha\beta\gamma}^{(n)} = (-1)^n \Lambda_{\alpha\beta\gamma}^{(n)}. \quad (29)$$

Covariant gauge fixing of this action is — at least formally — straightforward [14], so that covariant Green functions can be written down.

Of course we have yet to show that the action really describes a single chiral p-form. A naive argument which suggests that it does goes as follows: first use Gauss' Law to set the longitudinal degrees of freedom to zero. Then fix the remaining gauge freedom by the condition

$$E_{ab}^{(n)} + B_{ab}^{(n)} = 0 , \quad n > 0 . \quad (30)$$

Now all fields with $n > 0$ vanish as a consequence of the constraints, and the fields with $n = 0$ obey the same equations that we studied before, hence they do indeed describe a single chiral p-form. However, because of the occurrence of an infinite set of fields in the covariant action it is necessary to scrutinize this naive argument carefully to see that no inconsistencies arise. We remark in passing that the same action but with only a finite set of N fields describes either one self-dual and one anti-self-dual p-form, or a pair of self-dual p-forms one of which contributes with a negative sign to the total energy, depending on whether N is odd or even. Hence the infinite set can not be avoided. Incidentally this is similar to some covariant modifications of the Green- Schwarz type of action for superparticles [15].

To study the consistency of the covariant formulation we restrict ourselves to the original McClain-Wu-Yu action in $1 + 1$ dimensions [8];

$$S = \sum_{n=0}^{\infty} \int d^2x \left[\dot{\varphi}_{(n)} \pi_{(n)} - \frac{(-1)^n}{2} (\pi_{(n)}^2 + \varphi_{(n)}'^2) - \Lambda_{(n+1)} \Psi_{(n+1)} \right] , \quad (31)$$

where $\varphi_{(n)}$ are an infinite collection of scalar fields and $\pi_{(n)}$ their momenta, and the constraints are

$$\Psi_{(n+1)} = \pi_{(n)} - \varphi_{(n)}' + \pi_{(n+1)} + \varphi_{(n+1)}' \approx 0 . \quad (32)$$

It is easy to give an example of a formal calculation which leads to a contradiction [16]. Thus, define

$$\Pi \equiv \sum_{n=0}^{\infty} \pi_{(n)} . \quad (33)$$

This expression is formally gauge invariant. A formal addition of the constraints gives

$$\Pi = \frac{1}{2} (\pi_{(0)} + \varphi_{(0)}') + \frac{1}{2} \sum_{n=0}^{\infty} \Psi_{(n)} \approx \frac{1}{2} (\pi_{(0)} + \varphi_{(0)}') \equiv I_0 . \quad (34)$$

There should then be two equivalent ways of expressing our single gauge invariant degree of freedom. However,

$$\{I_0(x), I_0(y)\} = \frac{1}{2} \partial_x \delta(x, y) , \quad (35)$$

which is weakly non-zero, while the Poisson bracket $\{\Pi(x), \Pi(y)\} = 0$. This contradicts the general theorem that

$$\tilde{F} \approx F \quad \& \quad \tilde{G} \approx G \quad \Rightarrow \quad \{\tilde{F}, \tilde{G}\} \approx \{F, G\} , \quad (36)$$

so that the naive approach is clearly wrong.

If one tries to do the calculation in detail, one sees that the problem occurs because one encounters the infinite sum

$$A_{total} = A(1 - 1 + 1 - 1 + 1 - 1 + \dots) . \quad (37)$$

A similar sum appears when one makes a naive attempt to calculate the gravitational anomaly in the quantum version of the model. One can of course look for a regularization procedure that permits one to set this sum equal to the answer one desires, but we prefer to formulate the theory in such a way that the sum never occurs, and we will now proceed to do this for the classical theory.

To avoid the many fallacies that appear with an unlimited number of variables we clearly need to specify what kinds of functions $\varphi_{(n)}$, $\pi_{(n)}$ that are allowed in the phase space of the model. In order that

$$\mathcal{H} = \frac{1}{2} \int d^2x \sum_{n=0}^{\infty} (-1)^n (\pi_{(n)}^2 + \varphi_{(n)}'^2) = \frac{1}{4} \int d^2x \sum_{n=0}^{\infty} (-1)^n [(\pi_{(n)} + \varphi_{(n)}')^2 + (\pi_{(n)} - \varphi_{(n)}')^2] \quad (38)$$

be finite, we demand that the fields and momenta are limited from above, in the sense that there exist some continuous and differentiable functions f and h such that

$$\int d^2x [f^2(x) + h^2(x)] \quad (39)$$

is finite and well-defined, and

$$|\pi_{(n)}(x) + \varphi_{(n)}'(x)| \leq \frac{f(x)}{n}, \quad n > N_0 \quad (40)$$

$$|\pi_{(n)}(x) - \varphi_{(n)}'(x)| \leq \frac{h(x)}{n}, \quad n > N_0 \quad (41)$$

for all $x = |\vec{x}|$. If this is satisfied, the Hamiltonian is well-defined and finite, since I_0 is finite and

$$\int d^2x \sum_{n=1}^{\infty} (-1)^n (\pi_{(n)}^2 + \varphi_{(n)}'^2) \leq \frac{1}{2} \int d^2x \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} (h^2 + f^2) = \frac{\pi^2}{24} \int d^2x (h^2 + f^2). \quad (42)$$

The convergence property should be gauge invariant as well as time translation invariant. Consider the gauge transformations generated by the first class constraints,

$$\begin{aligned} \delta\varphi_{(n)} &= \{\varphi_{(n)}, \int \epsilon_{(n)} \Psi_{(n)}\} = \epsilon_{(n+1)} + \epsilon_{(n)} \\ \delta\pi_{(n)} &= \{\pi_{(n)}, \int \epsilon_{(n)} \Psi_{(n)}\} = \epsilon'_{(n)} - \epsilon'_{(n+1)} \end{aligned} \quad (43)$$

and denote the transformed momenta and fields by

$$\tilde{\pi}_{(n)} = \pi_{(n)} + \delta\pi_{(n)} \quad \text{and} \quad \tilde{\varphi}_{(n)} = \varphi_{(n)} + \delta\varphi_{(n)}.$$

The gauge transformed series

$$\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (\tilde{\pi}_{(n)}^2 + \tilde{\varphi}_{(n)}'^2)$$

is obviously convergent if we demand that the gauge parameters $\epsilon_{(n)}$ are such that

$$\epsilon'_{(n)} = \epsilon'_{(n)}(x) \leq \frac{k(x)}{n}, \quad n > N_0 \quad (44)$$

for some suitable function $k(x)$.

By introducing the restriction (44) on the gauge parameters, the inconsistency that is brought about by the naive approach in (34) - (37) is avoided, just as was desired.

We next inspect the convergence property under time translation by expressing the fields as solutions to the Klein-Gordon equation, viz.

$$\varphi_{(n)} = B_{(n)}(x+t) + G_{(n)}(x-t) \quad (45)$$

where B and G are continuous, differentiable functions with derivatives

$$\frac{\partial}{\partial y} B_{(n)}(y) = b_{(n)}(y) \quad \text{and} \quad \frac{\partial}{\partial y} G_{(n)}(y) = g_{(n)}(y).$$

Since $\dot{\varphi}_{(n)} = \pi_{(n)}$, this implies that $\pi_{(n)} + \varphi'_{(n)} = 2b_{(n)}(x+t)$ and $\pi_{(n)} - \varphi'_{(n)} = -2g_{(n)}(x+t)$. If (40) and (41) are satisfied at the initial time $t = 0$, we get that

$$|\pi_{(n)}(x) + \varphi'_{(n)}(x)| = |2b_{(n)}(x)| \leq \frac{f(x)}{n} \quad (46)$$

for all x . This however implies that

$$|\pi_{(n)}(x, t) + \varphi'_{(n)}(x, t)| = |2b_{(n)}(x+t)| \leq \frac{f(x+t)}{n} \quad (47)$$

for all t as well as for all x , because we can always rewrite the argument as $x+t = x'$. The same goes for

$$|\pi_{(n)}(x, t) - \varphi'_{(n)}(x, t)| = |2g_{(n)}(x-t)| \leq \frac{h(x-t)}{n}. \quad (48)$$

So in conclusion, if the Hamiltonian is initially well-defined and finite, this also holds at any later times.

We finally investigate the form of the Lagrange multipliers. If we choose the gauge fixing condition

$$\pi_{(n)} + \varphi'_{(n)} = 0, \quad n > 0$$

it follows that

$$\pi_{(n)} - \varphi'_{(n)} = 0, \quad \text{all } n,$$

and only one degree of freedom remains, represented e.g as $\varphi'_{(0)}$ or $\pi_{(0)} + \varphi'_{(0)}$, with the corresponding Hamiltonian

$$\mathcal{H} = \frac{1}{4}(\pi_{(0)} + \varphi'_{(0)})^2.$$

Now, to guarantee that the Hamiltonian satisfies the weak equality

$$\mathcal{H} = \frac{1}{2} \int d^2x \sum_{n=0}^{\infty} (-1)^n (\pi_{(n)}^2 + \varphi_{(n)}'^2) = \frac{1}{4}(\pi_{(0)} + \varphi'_{(0)})^2 - \sum_{n=1}^{\infty} \Lambda_{(n)} \Psi_{(n)} \approx \frac{1}{4}(\pi_{(0)} + \varphi'_{(0)})^2 \quad (49)$$

the gauge parameters must take the form

$$\Lambda_{(n)} = \frac{1}{4}(-1)^n [\pi_{(n)} + \varphi'_{(n)} - \pi_{(n-1)} + \varphi'_{(n-1)}]. \quad (50)$$

Since

$$|\pi_{(n)} + \varphi'_{(n)} - \pi_{(n-1)} + \varphi'_{(n-1)}| \leq |\pi_{(n)} + \varphi'_{(n)}| + |\varphi'_{(n-1)} - \pi_{(n-1)}| \leq \frac{f(x)}{n} + \frac{h(x)}{n},$$

$|\Lambda_{(n)}|$ is clearly limited by some quantity $\sim 1/n$, and therefore the $\Lambda_{(n)}$ belong to the set of gauge parameters defined in (44).

In conclusion, if we demand that the fields and momenta satisfy the conditions (40) and (41), and that the gauge parameters satisfy (44), we obtain a consistent covariant formulation of the action.

4. SPECIES DOUBLING

Since chiral p-forms give rise to gravitational anomalies [7], it is not surprising that they also exhibit species doubling if one attempts to define them on a lattice. Here we will present a direct demonstration of this fact for chiral bosons in two spacetime dimensions.

It seems worthwhile to do this here, since the only demonstration in the literature [6] makes use of an action that does not describe a single chiral boson, so that the argument becomes quite indirect.

To begin, a Lagrangian that describes an ordinary scalar boson on a discretized space (with time kept continuous) is

$$\mathcal{L} = \frac{a}{2} \sum_n (\dot{\varphi}_n^2 - (\frac{\Delta\varphi_{(n)}}{a})^2), \quad (51)$$

where a is the lattice spacing and n denotes the lattice sites, running over integer values $1 \leq n \leq N$ where N is an integer. To discuss the most general discretization possible it is convenient to introduce the Fourier transform

$$\Delta\varphi_{(n)} = \frac{1}{n} \sum_k \Delta\varphi_{(k)} e^{-2\pi i k n / N}. \quad (52)$$

We assume that for any $\Delta\varphi_{(n)}$, we can express $\Delta\varphi_{(k)}$ as

$$\Delta\varphi_{(k)} = -i f(k) \varphi_{(k)} \quad (53)$$

where $f(k)$ is some periodic function of k . This is then the most general discretization that we will consider. In order to ensure that the theory be local on the lattice, $f(k)$ must be continuous.

Simple examples of discretizations are the symmetrical discretization

$$\Delta\varphi_{(n)} = \frac{1}{2}(\varphi_{(n+1)} - \varphi_{(n-1)}) \quad \Leftrightarrow \quad f(k) = \sin \frac{2\pi k}{n}, \quad (54)$$

and the asymmetrical discretization

$$\Delta\varphi_{(n)} = \varphi_{(n+1)} - \varphi_{(n)} \quad \Leftrightarrow \quad f(k) = 2e^{-i\pi k/N} \sin \frac{\pi k}{n}. \quad (55)$$

As is well known (if not the reader will be reminded soon), the symmetrical discretization leads to species doubling in the continuum limit, while the asymmetrical one does not.

The Hamiltonian corresponding to (51) is

$$\begin{aligned} \mathcal{H} &= \frac{a}{2} \sum_n (\dot{\varphi}_{(n)}^2 + (\frac{\Delta\varphi_{(n)}}{a})^2) = \\ &= \frac{a}{2N} \sum_k (\dot{\varphi}_{(k)} \dot{\varphi}_{-k} - \frac{1}{a^2} \varphi_{(k)} \varphi_{-k} f(k) f(-k)) . \end{aligned} \quad (56)$$

In order to ensure that \mathcal{H} be real and positive, we have to require that

$$f(-k) = -f^*(k) , \quad (57)$$

which implies that

$$\mathcal{H} = \frac{a}{2N} \sum_k (\dot{\varphi}_{(k)} \dot{\varphi}_{-k} + \frac{1}{a^2} \varphi_{(k)} \varphi_{-k} f(k) f^*(k)) > 0 , \quad (58)$$

as desired.

We now add the self-duality requirement. This means that we impose the condition

$$\dot{\varphi}_n = \frac{1}{a} \Delta \varphi_n \quad \Leftrightarrow \quad \dot{\varphi}_k = -\frac{i}{a} f(k) \varphi_k . \quad (59)$$

However, the equation of motion that follows from our Lagrangian is, in momentum space,

$$a^2 \ddot{\varphi}_{(k)} - \varphi_{(k)} f(k) f(-k) = 0 . \quad (60)$$

Therefore the self-duality condition is consistent with the equations of motion if and only if

$$f(-k) = -f(k) . \quad (61)$$

If we refer back to eq. (57) we see that the self-duality requirement forces $f(k)$ to be a real and odd function. Since it is also periodic, it has to have an even number of zeroes. In particular, the asymmetrical discretization given above is not allowed by self-duality.

Now the point is that if we go to the continuum limit, each zero of the dispersion relation will contribute one degree of freedom to the continuum model, and the chirality of this degree of freedom will depend on the slope of $f(k)$ at its zero (see ref. [17]). For us, this implies that the continuum model will have an equal number of self-dual and anti-self dual degrees of freedom. This is species doubling.

5. OPEN QUESTIONS

In section 2 of this paper we tried to make a point which has not been made very clear in the literature, namely that the obstruction to manifest covariance usually is most severe for the Green functions (and by implication for the Poisson brackets). A manifestly covariant Lagrangian may well exist in a model which lacks manifestly covariant Green functions. In section 3 we discussed a proposed resolution of the problem for chiral p-forms, due to McClain, Wu and Yu. Their proposal involves an infinite set of gauge fields, and we drew attention to some difficulties that must be resolved. In a situation with an infinite number of fields, a straightforward naive treatment leads to expressions like $A_{total} = A(1 - 1 + 1 - 1 + 1 - 1 + \dots)$, where the sum depends upon the regularization procedure. This can however be avoided by using a more rigorous formulation. In the classical theory we showed that a rigorous formulation can indeed be given, provided that one restricts the phase space as well as the Lagrange multipliers that are allowed to smear the constraints suitably. In effect, one has to make a careful distinction between proper and improper constraints. We discussed only the case $D = 2$, but the generalization to

higher dimensions should not offer any difficulties. However, we have left the definition of the quantum theory as an open question.

In section 4 we showed directly how species doubling occurs on the lattice in $1 + 1$ dimensions. It would be of some interest to consider the general case, and to bring it up to the level of rigour that exists for chiral fermions [18], but we have not done so.

It is clear that there is a close connection between species doubling and the gravitational anomaly. Alvarez-Gaumé and Witten [7] argue that the latter is also closely connected to the lack of a manifestly covariant description. If the McClain-Wu-Yu proposal can be shown to work for the quantum theory with coupling to external fields included, then this argument will have to be scrutinized.

There are many interesting aspects of chiral p-forms that we have not touched at all. For instance, it has been suggested [19] that the difficulties with manifest covariance that we have encountered are akin to those that stand in the way of electrodynamics coupled to both electric and magnetic charges. The question whether an alternative formulation in terms of fermionic fields exists (in any twice odd dimension) is another issue that ought to be looked into. Finally, a thoughtful comparison of chiral p-forms with the chiral formulation of Einstein's equations due to Ashtekar [20] (in the twice even dimension $D = 4$) could give rise to valuable insights.

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